PROOF OF A CONJECTURE OF BÁRÁNY, KATCHALSKI AND PACH

MÁRTON NASZÓDI

Abstract. In [BKP84], Bárány, Katchalski and Pach proved the following quantitative form of Helly's theorem. If the intersection of a family of convex sets in $\mathbb{R}^d$ is of volume one, then the intersection of some subfamily of at most $2d$ members is of volume at most some constant $v(d)$. In [BKP82], the bound $v(d) \leq d^{2d^2}$ is proved and $v(d) \leq d^{cd}$ is conjectured. We confirm it.

1. Introduction and Preliminaries

Theorem 1.1. *Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$ such that the volume of its intersection is* $\mathrm{vol}\,(\cap\mathcal{F}) > 0$. *Then there is a subfamily $\mathcal{G}$ of $\mathcal{F}$ with $|\mathcal{G}| \leq 2d$ and* $\mathrm{vol}\,(\cap\mathcal{G}) \leq Ce^d d^{2d}\,\mathrm{vol}\,(\cap\mathcal{F})$, *where $C > 0$ is a universal constant.*

We recall the note from [BKP84] that the number $2d$ is optimal, as shown by the $2d$ half-spaces supporting the facets of the cube.

We introduce notations and tools that we will use in the proof. We denote the closed unit ball centered at the origin o in the d-dimensional Euclidean space $\mathbb{R}^d$ by $\mathbf{B}$. For the scalar product of $u, v \in \mathbb{R}^d$, we use $\langle u, v\rangle$, and the length of u is $|u| = \sqrt{\langle u, u\rangle}$. The tensor product $u \otimes u$ is the rank one linear operator that maps any $x \in \mathbb{R}^d$ to the vector $(u \otimes u)x = \langle u, x\rangle\, u \in \mathbb{R}^d$. For a set $A \subset \mathbb{R}^d$, we denote its polar by $A^* = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1, \text{ for all } x \in A\}$. The volume of a set is denoted by $\mathrm{vol}\,(\cdot)$.

Definition 1.2. We say that a set of vectors $w_1, \ldots, w_m \in \mathbb{R}^d$ with weights $c_1, \ldots, c_m > 0$ form a *John's decomposition of the identity*, if

$$\sum_{i=1}^{m} c_i w_i = o \quad \text{and} \quad \sum_{i=1}^{m} c_i w_i \otimes w_i = I, \tag{1}$$

where I is the identity operator on $\mathbb{R}^d$.

A *convex body* is a compact convex set in $\mathbb{R}^d$ with non-empty interior. We recall John's theorem [Joh48] (see also [Bal97]).

Lemma 1.3 (John's theorem)**.** *For any convex body K in $\mathbb{R}^d$, there is a unique ellipsoid of maximal volume in K. Furthermore, this ellipsoid is $\mathbf{B}$ if, and only if, there are points $w_1, \ldots, w_m \in \mathrm{bd}\,\mathbf{B} \cap \mathrm{bd}\,K$ (called* contact points*) and corresponding weights $c_1, \ldots, c_m > 0$ that form a John's decomposition of the identity.*

It is not difficult to see that if $w_1, \ldots, w_m \in \mathrm{bd}\,\mathbf{B}$ and corresponding weights $c_1, \ldots, c_m > 0$ form a John's decomposition of the identity, then $\{w_1, \ldots, w_m\}^* \subset d\mathbf{B}$, cf. [Bal97] or Theorem 5.1 in [GLMP04]. By polarity, we also obtain that $\frac{1}{d}\mathbf{B} \subset \mathrm{conv}(\{w_1, \ldots, w_m\})$.

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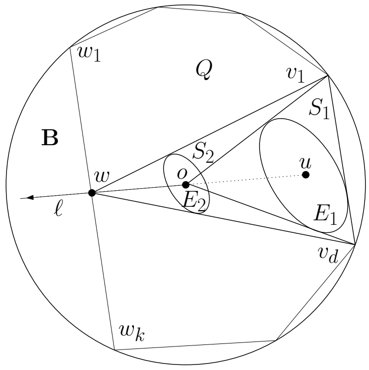

FIGURE 1.

One can verify that if Δ is a simplex in $\mathbb{R}^d$, and E is the largest volume ellipsoid in Δ, then

$$\frac{\mathrm{vol}\,(E)}{\mathrm{vol}\,(\Delta)} = \frac{d!\,\mathrm{vol}\,(\mathbf{B})}{d^{d/2}(d+1)^{(d+1)/2}}. \tag{2}$$

We will use the following form of the Dvoretzky-Rogers lemma [DR50].

Lemma 1.4 (Dvoretzky-Rogers lemma)**.** *Assume that* $w_1, \dots, w_m \in \mathrm{bd}\,\mathbf{B}$ *and* $c_1, \dots, c_m > 0$ *form a John's decomposition of the identity. Then there is an orthonormal basis* $z_1, \dots, z_d$ *of* $\mathbb{R}^d$*, and a subset* $\{v_1, \dots, v_d\}$ *of* $\{w_1, \dots, w_m\}$ *such that*

$$v_i \in \mathrm{span}\{z_1, \dots, z_i\}, \quad \textit{and} \quad \sqrt{\frac{d-i+1}{d}} \le \langle v_i, z_i \rangle \le 1, \quad \textit{for all } i = 1, \dots, d. \tag{3}$$

This lemma is usually stated in the setting of John's theorem, that is, when the vectors are contact points of a convex body K with its maximal volume ellipsoid, which is $\mathbf{B}$. And often, it is assumed in the statement that K is symmetric about the origin, see for example [BGVV14]. Since we make no such assumption (in fact, we make no reference to K in the statement of Lemma 1.4), we give a proof in Section 3.

2. Proof of Theorem 1.1

Without loss of generality, we may assume that $\mathcal{F}$ consists of closed half-spaces, and also that $\mathrm{vol}\,(\cap\mathcal{F}) < \infty$, that is, $\cap\mathcal{F}$ is a convex body in $\mathbb{R}^d$. As shown in [BKP84], by continuity, we may also assume that $\mathcal{F}$ is a finite family, that is $P = \cap\mathcal{F}$ is a d-dimensional polyhedron.

The problem is clearly affine invariant, so we may assume that $\mathbf{B} \subset P$ is the ellipsoid of maximal volume in P.

By Lemma 1.3, there are contact points $w_1, \dots, w_m \in \mathrm{bd}\,\mathbf{B} \cap \mathrm{bd}\,P$ (and weights $c_1, \dots, c_m > 0$) that form a John's decomposition of the identity. We denote their convex hull by $Q = \mathrm{conv}\{w_1. \dots, w_m\}$. Lemma 1.4 yields that there is an orthonormal basis $z_1, \dots, z_d$ of $\mathbb{R}^d$, and a subset $\{v_1, \dots, v_d\}$ of the contact points $\{w_1, \dots, w_m\}$ such that (3) holds.

Let $S_1 = \mathrm{conv}\{o, v_1, v_2, \dots, v_d\}$ be the simplex spanned by these contact points, and let E_1 be the largest volume ellipsoid contained in S_1. We denote the center of E_1 by u. Let ℓ be the ray emanating from the origin in the direction of the vector $-u$. Clearly, the origin is in the interior of Q. In fact, by the remark following Lemma 1.3, $\frac{1}{d}\mathbf{B} \subset Q$. Let w be the point of intersection of the ray ℓ with $\mathrm{bd}\,Q$. Then $|w| \ge 1/d$. Let S_2 denote the simplex $S_1 = \mathrm{conv}\{w, v_1, v_2, \dots, v_d\}$. See Figure 1.

We apply a contraction with center w and ratio $\lambda = \frac{|w|}{|w-u|}$ on E_1 to obtain the ellipsoid E_2. Clearly, E_2 is centered at the origin and is contained in S_2. Furthermore,

$$\lambda = \frac{|w|}{|u|+|w|} \geq \frac{|w|}{1+|w|} \geq \frac{1}{d+1}. \tag{4}$$

Since w is on $\operatorname{bd} Q$, by Caratheodory's theorem, w is in the convex hull of some set of at most d vertices of Q. By re-indexing the vertices, we may assume that $w \in \operatorname{conv}\{w_1, \ldots, w_k\}$ with $k \leq d$. Now,

$$E_2 \subset S_2 \subset \operatorname{conv}\{w_1, \ldots, w_k, v_1, \ldots, v_d\}. \tag{5}$$

Let $X = \{w_1, \ldots, w_k, v_1, \ldots, v_d\}$ be the set of these unit vectors, and let $\mathcal{G}$ denote the family of those half-space which support $\mathbf{B}$ at the points of X. Clearly, $|\mathcal{G}| \leq 2d$. Since the points of X are contact points of P and $\mathbf{B}$, we have that $\mathcal{G} \subseteq \mathcal{F}$. By (5),

$$\cap \mathcal{G} = X^* \subset E_2^*. \tag{6}$$

Since $\mathbf{B} \subset \cap \mathcal{F}$, by (6) and (4), and (2) we have

$$\frac{\operatorname{vol}(\cap\mathcal{G})}{\operatorname{vol}(\cap\mathcal{F})} \leq \frac{\operatorname{vol}(E_2^*)}{\operatorname{vol}(\mathbf{B})} = \frac{\operatorname{vol}(\mathbf{B})}{\operatorname{vol}(E_2)} \leq (d+1)^d \frac{\operatorname{vol}(\mathbf{B})}{\operatorname{vol}(E_1)} = \frac{d^{d/2}(d+1)^{(3d+1)/2}}{d! \operatorname{vol}(S_1)}. \tag{7}$$

By (3),

$$\operatorname{vol}(S_1) \geq \frac{1}{d!} \cdot \frac{\sqrt{d!}}{d^{d/2}} = \frac{1}{\sqrt{d!} d^{d/2}}, \tag{8}$$

which, combined with (7), yields the desired result, finishing the proof of Theorem 1.1.

Remark 2.1. In the proof, in place of the Dvoretzky-Rogers lemma, we could select the d vectors $v_1, \ldots, v_d$ from the contact points randomly: picking w_i with probability c_i/d for $i = 1, \ldots, m$, and repeating this picking independently d times. Pivovarov proved (cf. Lemma 3 in [Piv10]) that the expected volume of the random simplex S_1 obtained this way is the same as the right hand side in (8).

3. Proof of Lemma 1.4

We follow the proof in [BGVV14].

Claim 3.1. Assume that $w_1, \ldots, w_m \in \operatorname{bd} \mathbf{B}$ and $c_1, \ldots, c_m > 0$ form a John's decomposition of the identity. Then for any linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ there is an $\ell \in \{1, \ldots, m\}$ such that

$$\langle w_\ell, T w_\ell \rangle \geq \frac{\operatorname{tr} T}{d}, \tag{9}$$

where $\operatorname{tr} T$ denotes the trace of T.

For matrices $A, B \in \mathbb{R}^{d \times d}$ we use $\langle A, B \rangle = \operatorname{tr}\left(AB^T\right)$ to denote their Frobenius product. To prove the claim, we observe that

$$\frac{\operatorname{tr} T}{d} = \frac{1}{d}\langle T, I \rangle = \frac{1}{d}\sum_{i=1}^{m} c_i \langle T, w_i \otimes w_i \rangle = \frac{1}{d}\sum_{i=1}^{m} c_i \langle T w_i, w_i \rangle .$$

Since $\sum_{i=1}^m c_i = d$, the right hand side is a weighted average of the values $\langle T w_i, w_i \rangle$. Clearly, some value is at least the average, yielding Claim 3.1.

We define z_i and v_i inductively. First, let $z_1 = v_1 = w_1$. Assume that, for some $k < d$, we have found z_i and v_i, for all $i = 1, \ldots, k$. Let $F = \operatorname{span}\{z_1, \ldots, z_k\}$, and

let T be the orthogonal projection onto the orthogonal complement $F^\perp$ of F. Clearly, $\operatorname{tr} T = \dim F^\perp = d-k$. By Claim 3.1, for some $\ell \in \{1, \dots, m\}$ we have

$$|Tw_\ell|^2 = \langle Tw_\ell, w_\ell \rangle \geq \frac{d-k}{d}.$$

Let $v_{k+1} = w_\ell$ and $z_{k+1} = \frac{Tw_\ell}{|Tw_\ell|}$. Clearly, $v_{k+1} \in \operatorname{span}\{z_1, \dots, z_{k+1}\}$. Moreover,

$$\langle v_{k+1}, z_{k+1} \rangle = \frac{\langle Tw_\ell, w_\ell \rangle}{|Tw_\ell|} = \frac{|Tw_\ell|^2}{|Tw_\ell|} = |Tw_\ell| \geq \sqrt{\frac{d-k}{d}},$$

finishing the proof of Lemma 1.4.

Note that in this proof, we did not use the fact that, in a John's decomposition of the identity, the vectors are balanced, that is $\sum_{i=1}^m c_i w_i = o$.

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Márton Naszódi, EPFL, SB MATHGEOM DCG CH-1015 Lausanne Switzerland, and ELTE, Dept. of Geometry, Lorand Eötvös University, Pázmány Péter Sétány 1/C Budapest, Hungary 1117

E-mail address: marton.naszodi@math.elte.hu